MATHEMATICS

IMBEDDING A FINITE METRIC SET IN AN N-DIMENSIONAL MINKOWSKI SPACE [1)]

BY

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The Minkowski space in which the unit sphere is a hypercube is particularly useful for the study of finite sets of points. It is well known that any separable metric space can be imbedded isometrically in the space m of bounded sequences of real numbers with $x=\{\xi_j\}$, $y=\{\eta_j\}$, and distance $xy=\limsup_j |\xi_j-\eta_j|$. [See 1, p. 187]. By the same construction, due to FRÉCHET, a finite set of $n+1$ points can be imbedded in an n-dimensional subspace of m. [2, p. 535].

Here we shall use a different construction to imbed a set of $n+2$ points ($n \geqslant 2$) in an n-dimensional subspace of m. Our construction will show also that any minimal point of the set can be imbedded in the same subspace.

We shall start with a set $S=\{p_1, \ldots, p_n\}$ with distances $p_i p_j$ defined, satisfying the usual properties for a metric. We shall define M_n as the set of points $x=(x_1, \ldots, x_n)$ with distance function $xy=\max_j |x_j-y_j|$. A minimal point of S is a point p_0 such that $S \cup \{p_0\}$ is metric and $\sum_{i=1}^{n} p_0 p_i = \min \sum_{i=1}^{n} p p_i$ where the minimum is over all points p such that $S \cup \{p\}$ is metric.

In a forthcoming paper [3] we have shown that p_0 must separate pairs of points of S in the following way:

Given any permutation of the first n numbers,

$$\pi=\begin{pmatrix} 1 & 2 & \ldots & n \\ i_i & i_2 & \ldots & i_n \end{pmatrix},\ f(\pi)=\sum_{\nu=1}^{n} p_\nu p_{i_\nu}$$

is a sum of n linked distances. (We are indebted to Professor A. M. OSTROWSKI for our notation.) The separating point theorem states that if we choose π so that the sum of n linked distances is a maximum, there

[1)] This paper is a revision of part of the author's doctoral dissertation at the University of Pennsylvania, prepared under the direction of Professor I. J. Schoenberg.

exists a point p such that $S \cup \{p\}$ is metric and $pp_\nu + pp_{i_\nu} = p_\nu p_{i_\nu}$ for every ν. Such p is, of course, a minimal point, and therefore $\sum_{i=1}^{n} p_0 p_i$ is exactly one-half the maximum sum of n linked distances. The concept of a separating point seems particularly appropriate in studying M_n.

Imbedding $n+2$ points in M_n.

Theorem 1. Any metric set of $n+2$ points, where $n \geqslant 2$, can be imbedded isometrically in M_n.

Proof. We use an induction argument: the following lemma settles the question for $n=2$.

Lemma 1. If $S = \{p_1, p_2, p_3, p_4\}$ is a metric set containing exactly 4 points and if p_0 is any separating point of S, then S and p_0 can be imbedded isometrically in M_2.

Proof. If a set contains only 4 points, the maximal sum of 4 linked distances is twice the maximum sum of two disjoint distances. Let $p_1p_2 + p_3p_4$ be a maximal sum of disjoint distances, with $p_0p_1 + p_0p_2 = p_1p_2$ and $p_0p_3 + p_0p_4 = p_3p_4$. The mapping is as follows:

$$p_0 \to (0, 0)$$
$$p_1 \to (-p_0p_1,\ p_0p_4 - p_1p_4)$$
$$p_2 \to (p_0p_2,\ p_2p_3 - p_0p_3)$$
$$p_3 \to (p_1p_3 - p_0p_1,\ -p_0p_3)$$
$$p_4 \to (p_0p_2 - p_2p_4,\ p_0p_4)$$

We must show that if

$$p_h \to (x_{1h}, x_{2h})$$
$$p_i \to (x_{1i}, x_{2i})$$

then $|x_{jh} - x_{ji}| \leqslant p_h p_i$ for $j = 1, 2$, with equality for at least one value of j.

The equalities are all obvious, as are the inequalities involving p_0. For the others:

$$p_1p_2\colon |p_2p_3 - p_0p_3 - p_0p_4 + p_1p_4| = |p_2p_3 - p_3p_4 + p_1p_4|$$

We have both $p_2p_3 - p_3p_4 + p_1p_4 \leqslant p_1p_2$ and $-p_2p_3 + p_3p_4 - p_1p_4 \leqslant -$ $-p_2p_3 + p_3p_1 \leqslant p_1p_2$

$$p_1p_3\colon |-p_0p_3 - p_0p_4 + p_1p_4| = |-p_3p_4 + p_1p_4| \leqslant p_1p_3$$
$$p_1p_4\colon |p_0p_2 - p_2p_4 + p_0p_1| = |p_1p_2 - p_2p_4| \leqslant p_1p_4$$
$$p_2p_3\colon |p_1p_3 - p_0p_1 - p_0p_2| = |p_1p_3 - p_1p_2| \leqslant p_2p_3$$
$$p_2p_4\colon |p_0p_4 - p_2p_3 + p_0p_3| = |p_3p_4 - p_2p_3| \leqslant p_2p_4$$
$$p_3p_4\colon |p_1p_3 - p_0p_1 - p_0p_2 + p_2p_4| = |p_1p_3 - p_1p_2 + p_2p_4|$$

and we use the reasoning we used for p_1p_2.

To complete the induction proof of Theorem 1, we assume that $n+1$ points can be imbedded in M_{n-1}. Lemma 2 furnishes a construction for imbedding $n+2$ points in M_n:

Lemma 2. Given a metric set of $m+1$ points, if some m of these can be imbedded isometrically in M_k, then the $m+1$ points can be imbedded isometrically in M_{k+1}.

Proof. Let $p_h \to (x_{1h}, \ldots, x_{kh})$ be the given mapping, with $h=1, \ldots, m$. We map the m points into M_{k+1}:

$$p_h \to (x_{1h}, \ldots, x_{kh}, p_h p_{m+1})$$

No distances are changed, since $|p_h p_{m+1} - p_i p_{m+1}| \leqslant p_h p_i$. Suppose, now, we map $p_{m+1} \to (x_1, \ldots, x_k, 0)$, with $x_1, \ldots, x_k$ real numbers to be determined. If there exist k numbers such that for $j=1, \ldots, k$

$$(1) \qquad |x_j - x_{jh}| \leqslant p_h p_{m+1} \text{ for all } h$$

distances $p_h p_{m+1}$ will be preserved, and we'll be finished.

Rewriting (1) we get: $x_j - x_{jh} \leqslant p_h p_{m+1}$ and $x_{jh} - x_j \leqslant p_h p_{m+1}$ or

$$(2) \qquad x_{jh} - p_h p_{m+1} \leqslant x_j \leqslant p_h p_{m+1} + x_{jh}.$$

Given any point p_h, the left and right inequalities of (2) define a closed segment of the real line on which x_j may lie. Given any other point, say p_i, another closed line segment is determined. These two segments must intersect if $x_{jh} - p_h p_{m+1} \leqslant p_i p_{m+1} + x_{ji}$ for $1 \leqslant h,\ i \leqslant m$.

But we know $x_{jh} - x_{ji} \leqslant p_h p_i \leqslant p_h p_{m+1} + p_i p_{m+1}$ because p_h and p_i were imbedded in M_k with x_{jh} and x_{ji} their respective j^{th} coordinates.

Thus we have m closed line segments on E_1, each defined by (2), with every two of them intersecting; all m segments intersect, by Helly's theorem. (Of course, Helly's theorem is elementary in one dimension.) x_j can be chosen as a point in the intersection.

Imbedding a minimal point.

We can use Lemma 2 for an imbedding theorem concerning the minimal point:

Theorem 2. If a finite metric set can be imbedded isometrically in M_k, so also can a minimal point p_0.

Proof. Let $\{p_1, \ldots, p_n\}$ be a metric set and p_0 be a minimal point. The n given points can be imbedded in M_k and these plus p_0 in M_{k+1} by the method of Lemma 2. Now we consider the projection of these $n+1$ points on M_k. If we use the first k coordinates of the mapping, we get back $\{p_1, \ldots, p_n\}$, of course. Let the projection of the image of p_0 on M_k be $p_m = (x_1, \ldots, x_k)$. But for $j=1, \ldots, k$ and $i=1, \ldots, n$, $|x_j - x_{ji}| \leqslant p_i p_0$. Hence $p_m p_i \leqslant p_0 p_i$ and $\sum_{i=1}^{n} p_m p_i \leqslant \sum_{i=1}^{n} p_0 p_i$. But p_0 was a minimal point.

Therefore these two sums are equal. This and $p_m p_i \leqslant p_0 p_i$ for all i imply $p_m p_i = p_0 p_i$, which shows that $p_m (\varepsilon M_k)$ is the desired image of the minimal point.

Our work thus far suggests that we exploit the geometry in constructing a minimal point for a set S. If S is contained in M_2, this is particularly rewarding. An elegant construction by Professor SCHOENBERG gives us all separating (i.e., minimal) points. First, suppose there are an even number of points in S – say $2m$. Then the set of separating points is the set of points lying in a rectangle constructed as follows:

Consider in the xy plane the lines of the equation $y = x + s$ such that m of the points are above the line and m of the points are below it. Call $y = x + s_1$ the lower bound of such lines. (Clearly at least m points are above or on this line, and at least m points are below or on it.) Call $y = x + s_2$ the higher bound ($s_1 \leqslant s_2$).

Likewise, consider the lines $y = -x + t$ with the same separation property

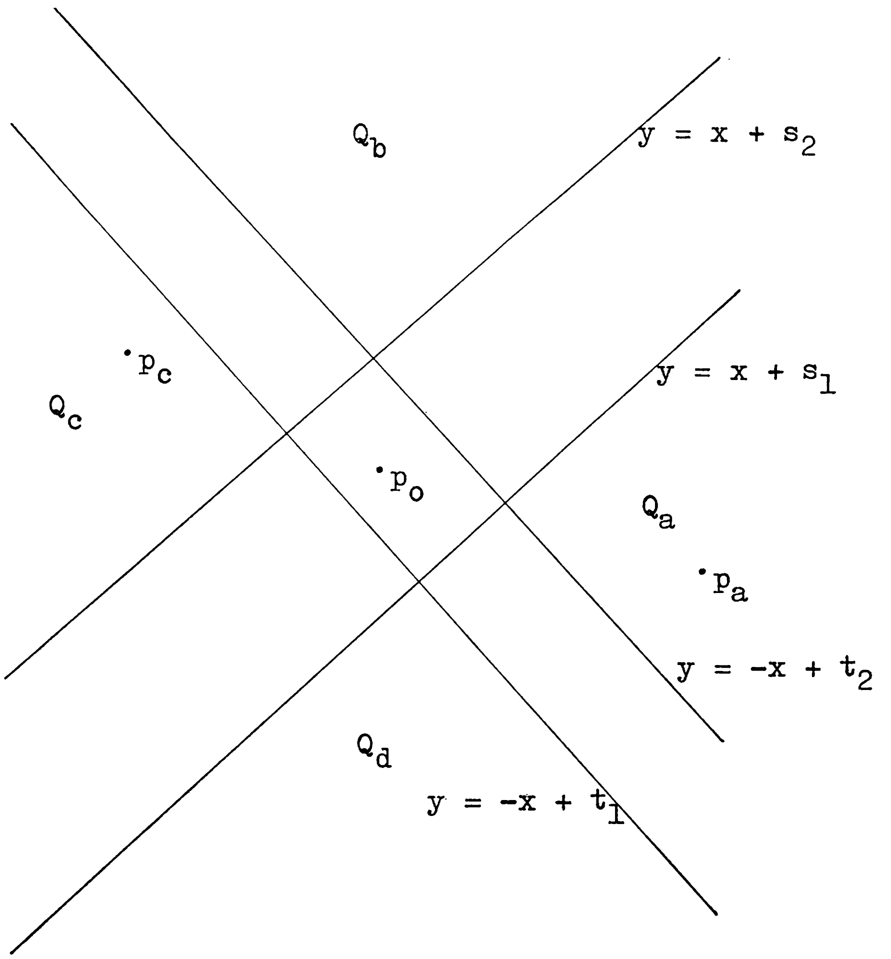

Fig. 1

as above, and call $y=-x+t_1$ the lower bound and $y=-x+t_2$ the higher bound $(t_1 \leqslant t_2)$.

The rectangle whose sides are the above lines is the locus of separating points in M_2. For consider the four closed quadrants Q_a, Q_b, Q_c, Q_d obtained by removing the two parallel strips. Call the number of points in these quadrants n_a, n_b, n_c, n_d. Then $n_a=n_c$ and $n_b=n_d$. The points in Q_a can be paired off in any way we like with those in Q_c and the points in Q_b with those in Q_d.

Then if p_0 lies in the closed rectangle p_0 is a separating point. For example, let $p_a=(x_a, y_a)$ be any point in Q_a; $p_c=(x_c, y_c)$ be any point in Q_c; and $p_0=(x_0, y_0)$ be any point in the rectangle. Then $p_a p_c = x_a - x_c$: since $y_a \geqslant -x_a+t_2$ and $y_c \leqslant -x_c+t_1$, it follows that $x_a-x_c \geqslant -y_a+t_2+y_c-t_1 \geqslant \geqslant y_c-y_a$; and similarly $x_a-x_c \geqslant y_a-s_1-y_c+s_2 \geqslant y_a-y_c$. Also $p_0p_a=x_a-x_0$ and $p_0p_c=x_0-x_c$. Thus $p_0p_a+p_0p_c=p_ap_c$. In the same way, if p_b and p_d are points of Q_b and Q_d respectively, $p_0p_b+p_0p_d=p_bp_d$.

If S contains an odd number of points, a similar construction gives us not four lines and a rectangle but two lines intersecting in a single point. This point is the unique separating point of S.

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